Part I
Background

Background

Contents

1 Temporal Reasoning Systems

Information

Information: Any organized collection of symbols or signs produced:

- either by **observing** natural or artificial phenomena
- or by the **cognitive activity of agents**

useful for:

- understanding our world
- support decision-making
- communicate with other agents

Knowledge Representation and Reasoning: Theories and methods whose aim is to exploit all types of available information useful for problem solving and communication using intelligent machines

Temporal Reasoning

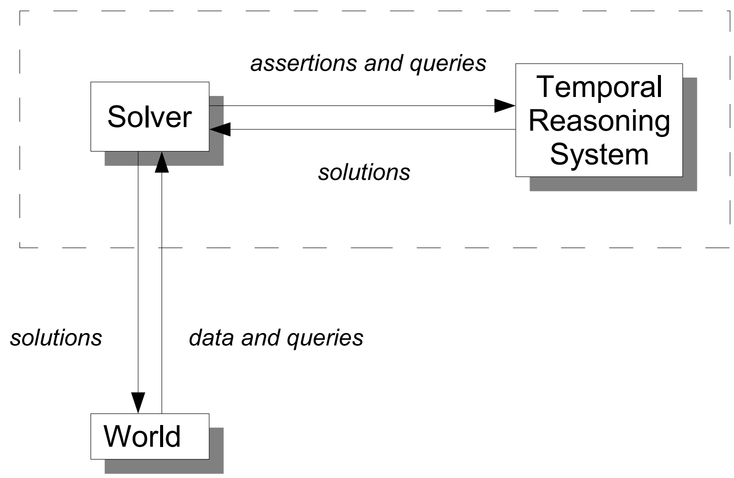
Time is an important aspect to be accounted for:

- real world is dynamic
- perceptions and human actions are characterized by time

Applications

- Medical diagnoses: *which disease presents this sequence of symptoms?*
- Planning: *which temporal relation exists between the actions A and B?*
- Temporal Databases: *which is the chronological order of a set of vases?*

Diagram of a Temporal Reasoner



Queries

A Temporal Reasoner should be able to answer to queries about temporal information, for example:

1. Is the information coherent? Which is a consistent scenario?
2. Can the event X_i happen between t_1 and t_2 instants after X_j?
3. Must the event X_i happen t instants before X_i?
4. In which instants t can the event X_i be verified?
5. If the event X_i happens in t_1 in which instants t_2 can X_j happen?

Representing time

If we want to take into account the time we have to consider several aspects

- **ontology:** how we can model time?
- **representation:** which hypotheses hold?
- **reasoning methods:** which entities allow obtaining the data in which we are interested?
- **algorithms:** efficiency - expressiveness

Temporal Logics

Three main approaches have been proposed to deal with time:

- Logics with temporal parameters
- modal temporal Logics [Prior57]
  - $P\Phi$: Φ was true
  - $F\Phi$: Φ will be true
- Reified Logics
  - Interval algebra (IA) [Allen83]
  - Event Calculus [Kowalski86]

$$HoldsAt(hand\_tool(box), t_1)$$

Temporal Reasoning using $CSPs$

A CSP is defined as a tuple $< X, D, R >$ where:

1. X is a set of variables $\{x_1, \ldots, x_n\}$
2. D is a finite set $\{d_1, \ldots, d_n\}$ of values such that $x_i \in D_i$
3. R is a set of relations $\{R_1, \ldots, R_k\}$ which specify the values d allowed by the constraints themselves

Solutions of a CSP

A solution of a CSP is an assignment of the variables which simultaneously satisfies all the constraints

It is possible to:

- check the existence of a solution
- search all the solutions
- search the optimal solution.

If at least a solution exists then the CSP is said satisfiable or **consistent**
The intersection of all the solutions gives the **minimal network**

Answering the queries

By representing the temporal problem using $CSPs$ the previous queries can be answered

1. the information is coherent: *check the consistency of the network*
2. event X_i can happen between t_1 and t_2 instants after X_j: *add the constraint $X_j - X_i \in [t_1, t_2]$ the network and check consistency*
3. event X_i must happen t instants before X_i: *assert the negation of that constraint and check the consistency*
4. In which instants t can the event X_i be verified? *the allowed instants are the minimal domain of $X_i - X_0$*
5. If the event X_i happens in t_1 in which instants t_2 can X_j happen? *check that $t_2 - t_1 \in X_j - X_i$ (minimal network)*

2 Temporal Information

Metric and qualitative temporal information

Two types of temporal information exist:

- qualitative information (relations)

"event A can happen before or during event B"

- metric information (numeric data)

"from 10:30 to 11 p.m."

2.1 Qualitative Temporal Information

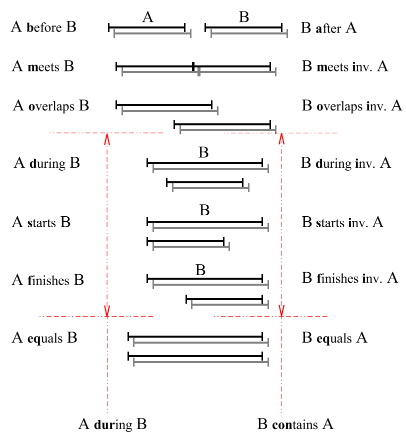
Allen's Interval Algebra (IA)

Allen's Interval Algebra is a qualitative temporal algebra based on 13 atomic relations:

- mutually exclusive
- jointly exhaustive

Example of relation

$$A\{b, m\}B$$



Operations in IA

A relational algebra is a set of relations closed under certain operations:

Allen's Interval Algebra is closed under

- inversion
$$(A\{b,m\}B)^{-1} = B\{bi,mi\}A$$
- intersection
$$A\{b,m\}B \cap A\{b\}B = A\{b\}B$$
- composition
$$A\{b,m\}B \circ B\{b\}C = A\{\{b \circ b\} \cup \{m \circ b\}\}C$$

Transitivity table of IA

Composition of atomic relations is given by a transitivity table

$\circ$	**b**	**a**	**d**	**di**	**o**	...
b	b	?	b d o m s	b	b	...
a	?	a	a d oi mi f	a	a d oi mi f	...
d	b	a	d	?	b d o m s	...
di	b di o m fi	a di oi mi si	o oi eq dur c	di	di o fi	...
...	...	...	...	...	...	...

Point Algebra (PA)

If events are points, only three relations are possible:

$$\{<, =, >\}$$

The operations defined are the same as IA

Example of relation

$$A\{<, =\}B$$

$\circ$	$<$	$>$	$=$
$<$	$<$	?	$<$
$>$	?	$>$	$>$
$=$	$<$	$>$	$=$

Table 1: Transitivity table for PA relations

Convex relations of qualitative algebras

If a network has only convex relations it can be minimized using Path Consistency algorithm:

- $\mathrm{PA}_c = PA\backslash\{\{<, >\}\}$
- the maximal tractable subalgebra of IA, called $\mathcal{H}$ has been identified by Nebel; it is formed by convex relations

Qualitative Algebra (QA)

The Qualitative Algebra between points and intervals is given by the union of:

- Allen's Interval Algebra
- the Point Algebra PA
- a set of 5 relations between Points and Intervals (PI relations)

$$\{b, a, d, s, f\}$$

Transitivity table of QA

Composition in QA involves transitivity tables for all the allowed combinations of relations (some do not have sense and are marked with $\oslash$, e.g. PI $\circ$ PI)

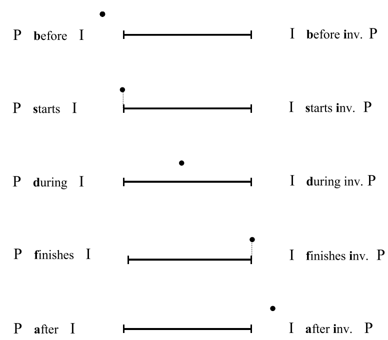


Figure 1: PI relations and their inverses

$\circ$	PP	PI	IP	II
PP	T_{PA}	T_1	$\oslash$	$\oslash$
PI	$\oslash$	$\oslash$	T_2	T_4
IP	T_1^T	T_3	$\oslash$	$\oslash$
II	$\oslash$	$\oslash$	T_4^T	T_{IA}

Table 2: Transitivity table of QA

2.2 Metric Temporal Information

Simple Temporal Problems ($STPs$)

A Simple Temporal Problem is defined as a tuple $< V, E >$ where:

- V is a set of variables $\{v_1, \ldots, v_n\}$ representing timepoints
- E is a set of constraints $\{e_1, \ldots, e_r\}$ between the variables in V

A constraint has the form

$$v_i[a, b]v_j$$

and means $a \leq v_j - v_i \leq b$

Operations in $STPs$

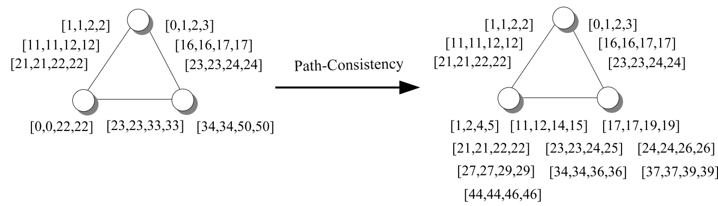
There are three fundamental operations:

1. inversion
$$(v_i[a, b]v_j)^{-1} = v_j[-b, -a]v_i$$
2. intersection
$$(v_i[a, b]v_j) \cap (v_i[c, d]v_j)$$
$$= (v_i[a, b] \cap [c, d]v_j)$$
3. composition
$$(v_i[a, b]v_j) \circ (v[c, d]v_j)$$
$$= (v[a + c, b + d]v_j)$$

The origin of complexity

Temporal problems are in general $\mathcal{NP}$-complete:

- complexity in metric constraints is due to fragmentation

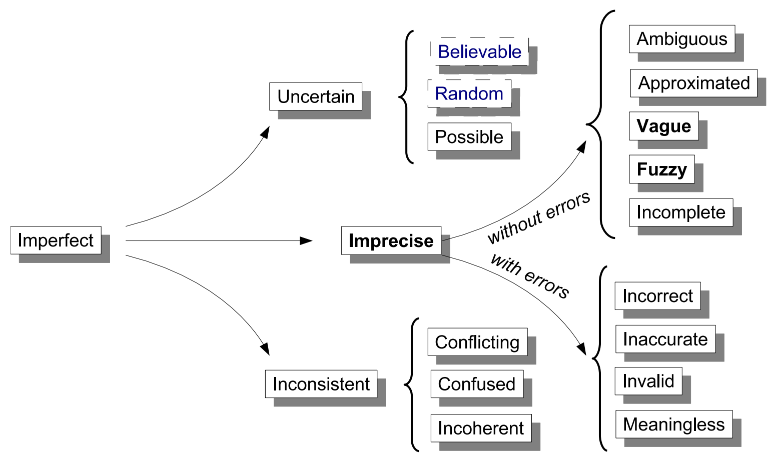


- complexity in qualitative constraints is intrinsic in the algebra

3 Imperfect data

3.1 Types of imperfections

Real data are imperfect data



The nature of uncertainty

Uncertainty is a property of the belief state of an agent

For example a robot has to grasp a block:

- "the block is on the table" is an **imprecise** fact
- "the block is near the centre of the table" is a **vague** fact
- "yesterday the block was in (10,12)" is an **unreliable** fact

3.2 Possibility Theory

Linguistic Gradual Information

Categories manipulated in natural language are not always all-or-nothing:

- "*Many* Americans are *tall*"
- John and Paul have *approximately* the same age"

Crisp sets are not sufficient!

The set of *young* ages is ill-defined, vague
Vague predicates: they have not a crisp boundary

Gradual truth

A proposition involving a gradual predicate can be true to a degree: a bottle can be neither empty not full, a 50-year old person is old to some extent

$$Truth(Old(Paul)) \in (0, 1)$$

Degrees of truth can be linguistic: "somewhat old", "rather old", "very old"

Forms of graduality

The existence of gradual predicates is due to

- matching a continuous observable scale and a finite vocabulary

  $$[0, 200]cm \to \{short, medium, tall\}$$

  there is no infinitely precise height s^* such that if $s > s^*$ $tall(s)$ is true otherwise s is false

  It is not that this threshold is unknown: it simply does not exist. The truth scale is continuous because the observable is continuous

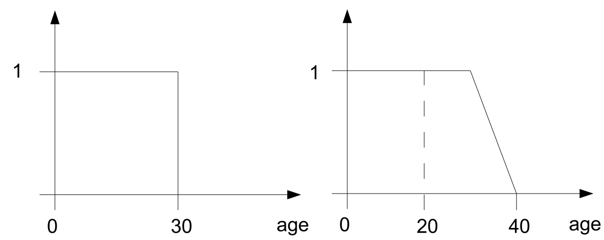
- The notion of typicality: elements of a class of objects can be more or less typical of that class: bird, chair, ...

  An ordering typicality relation: $x >_F y$ means x is more typically F than y

Fuzzy sets

Fuzzy set F on S: $\forall s, \mu_F(s) \in [0, 1]$
For example: F = *young*

A gradual representation preserves continuity and is less sensitive to the choice of a threshold



Definitions of a Fuzzy Set

1. A Fuzzy Set F is a set with gradual boundaries; can be defined using a generalized characteristic function (called membership function)

$$\mu_F : U \rightarrow [0, 1]$$

2. Equivalently, also as a weighted nested family of sets

$$F = \bigcup_{\alpha \in [0,1]} F_\alpha$$

where $F_\alpha = \{u : \mu_F(u) \geq \alpha\}$

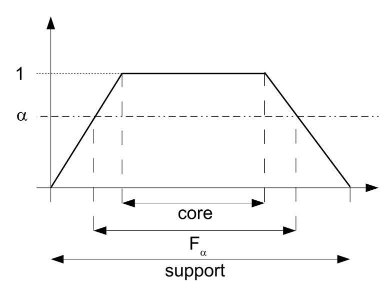
Fuzzy Interval

A membership function is a fuzzy interval

$$core(F) = \{u : \mu_F(u) = 1\}$$

$$support(F) = \{u : \mu_F(u) > 0\}$$

- the core includes most typical elements
- the support includes least typical elements



Motivation

The Possibility Theory allows working with qualitative models:

- it is more robust for modelling uncertain data (eg missing statistics)
- symbolic knowledge and numerical imprecision can be described
- it is a generalization of crisp Classical Logics, therefore it can represent also precise data when available

3.3 Possibility vs. Probability

Possibility vs. Probability

Probability Measure	**Membership Function**
Calculates the probability that an ill-known variable x ranging on U hits the well-known set A	Calculates the membership of a well-known variable x ranging on U hits the ill-known set A
Before an event happens	After it has happened
Measure Theory	Set Theory
Domain is 2^U (Boolean Algebra)	Domain is $[0,1] * U$ (Cannot be a Boolean Algebra)

Limits of classical $CSPs$

Classical $CSPs$:

- are rigid, since all constraints must be satisfied
- assign the same importance to all the constraints
- cannot specify uncertainty in the constraints

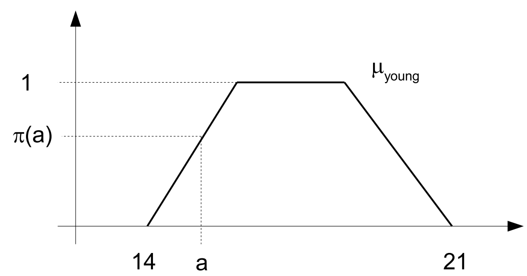
Fuzzy $CSPs$

It is possible to map a well-ordered partition $(E_1, E_2, \ldots, E_n)$ of constraints to a plausibility scale L using a possibility distribution π

A possibility distribution π_x is the representation of a state of knowledge: what an agent knows of the state of affairs x is

Conventions

- $\pi_x(s) = 0 \Leftrightarrow x = s$ is impossible, totally excluded (not expressible with $\geq_\pi$



- $\pi_x(s) = 1 \Leftrightarrow x = s$ is expected, normal, fully plausible, unsurprising
- $\pi_x(s) > \pi_x(s') \Leftrightarrow x = s$ more plausible than $x = s$

Example

Given the sentence "John is young"
$\pi_{young}(a)$ is the possibility that the age of John is a

Part II
From crisp to fuzzy constraint networks

Motivation

CSP is a general framework:

- A set of variables
- A set of constraints
- Major tasks: consistency checking, finding a solution, compute the minimal network

Temporal reasoning: specialized constraint-based reasoning frameworks

- Specific variable domains
- Restricted shape for constraints
- Specific properties and algorithms can be exploited to solve major tasks

Motivation (2)

FCSP is a generalization of classical CSP in order to reason with fuzzy constraints:

- A set of variables

- A set of fuzzy constraints
- Major tasks: determining the *consistency degree*, finding an *optimal solution*, compute the *minimal network*
- Links and similarities between CSP and FCSP at a general level

Idea: to what extent properties, theorems and algorithms of *specific frameworks* can be generalized to corresponding fuzzy frameworks?

Fuzzy constraint network (FCN)

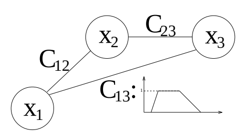
$\mathcal{N} = \langle X, D, C \rangle$ where:

- $X = \{x_1, \ldots, x_n\}$ (a set of variables)
- $D = \{D_1, \ldots, D_n\}$ (the set of relevant domains)
- $C = \{C_1, \ldots, C_m\}$ (a set of constraints)

where each constraint has the form $C_i = \langle V(C_i), R_i \rangle$, with

- $V(C_i) = \{y_1, \ldots, y_k\} \subseteq X$
- $R_i : D'_1 \times \ldots D'_k \to [0, 1]$

A graphical representation for binary networks (example)



Fuzzy constraint network (FCN)

Degree of local consistency Given $\overline{d}$ (instantiation of a set of variables $\mathcal{Y} \subseteq X$):

$$\mathrm{cons}(\overline{d}) = \min_{R_i | V(C_i) \subseteq \mathcal{Y}} R_i(\overline{d}^{\downarrow V(C_i)})$$

Solutions of $\mathcal{N}$ Complete instantiations $\overline{d}$ of the variables, with *consistency degree*

$$\deg(\overline{d}) = \mathrm{cons}(\overline{d})$$

Solution set of $\mathcal{N}$: the fuzzy set

$$\mathrm{SOL}(\mathcal{N}) : D_1 \times \ldots D_n \to [0, 1]$$

Fuzzy constraint network (FCN)

- *Consistency degree* of a network $\mathcal{N}$: consistency degree of the "best" solutions:

$$\sup_{\overline{d} \in D_1 \times \ldots \times D_n} \deg(\overline{d})$$

- *Optimal solutions* of $\mathcal{N}$: solutions $\overline{d}$ such as $\deg(\overline{d})$ is equal to the consistency degree of $\mathcal{N}$
- *Equivalence* of fuzzy constraint networks: the same variables, the same domanis, the same solution set

Constraint propagation algorithms, k-consistency and minimality

- Constraint propagation algorihms: maintain network equivalence, enforce local consistency of the network
- k-consistency:

$$\begin{aligned} \forall Y = \{y_1, \ldots, y_{k-1}\} \subseteq X, \forall y_k \in X \text{ with } y_k \notin Y, \\ \forall \overline{d} \in D'_1 \times \ldots \times D'_{k-1}, \\ \exists d_k \in D'_k \text{ such as } \mathrm{cons}(\overline{d}d_k) = \mathrm{cons}(\overline{d}) \end{aligned}$$

- Example: *path-consistency* is 2-consistency
- The *minimal network* is the "most explicit" one:

$$\begin{aligned} \forall \{x_i, x_j\} \subseteq X, \forall \overline{d'} \in D_i \times D_j, \\ \exists \overline{d} \in D_1 \times \ldots \times D_n \text{ such as } \mathrm{cons}(\overline{d}) = \mathrm{cons}(\overline{d'}) \end{aligned}$$

Relationship between classical and fuzzy constraint networks

A classical crisp constraint network can be seen as a fuzzy constraint network with preference degrees in $\{0, 1\}$ only.

Classical network	**Fuzzy network**
Preference degrees: $\{0, 1\}$	Preference degrees: $[0, 1]$
Consistency	Consistency degree
Solution	Optimal solution
k-consistency and minimality: extend a consistent instantiation to a consistent instantiation	k-consistency and minimality: extend an instantiation preserving its consistency degreee

Constraint-based reasoning frameworks

Scenarios of interest represented by means of (fuzzy or crisp) constraint networks

Definition 1. Class of crisp (fuzzy) constraint networks $\mathcal{HN}$ ($\mathcal{FN}$): a possibly infinite set of crisp (fuzzy) constraint networks.

TCSP: temporal constraint satisfaction problem

- variables: time points
- domains: $\Re$
- constraints: binary, $C_{ij} : x_i - x_j \in I, I = \{(a_1, b_1), \ldots (a_n, b_n)\}$

this can be indicated as $\mathcal{HN}_{TCSP}$

Constraint-based reasoning frameworks (2)

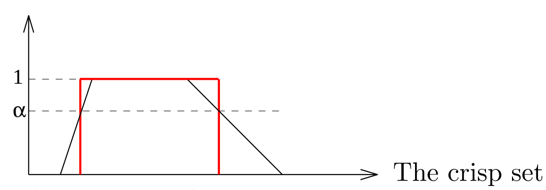
Allen's interval algebra (IA)

- variables: time intervals
- domains: $\Re^2$
- constraints: binary, disjunctions of 13 basic relations

this can be indicated as $\mathcal{FN}_{IA}$

A bridge between crisp and fuzzy reasoning frameworks

- **α-cut of a fuzzy set** R^{fuz}:



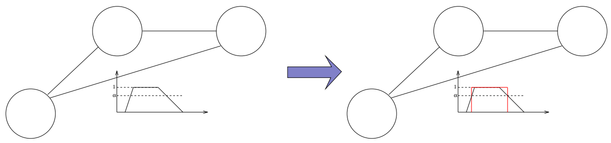
$R^{fuz}{}_{\alpha} = \{\overline{d} \mid R^{fuz}(\overline{d}) \geq \alpha\}$

- **α-cut of a fuzzy constraint** $C^{fuz} = \langle V, R^{fuz} \rangle$:

  The crisp constraint $C^{fuz}_{\alpha} = \langle V, R^{fuz}_{\alpha} \rangle$

A bridge between crisp and fuzzy reasoning frameworks (2)

- **α-cut of a fuzzy constraint network $\mathcal{N}^{fuz}$:**

  The crisp constraint network $\mathcal{N}^{fuz}{}_{\alpha} = \langle X, D, \{C_{1\alpha}, \ldots, C_{m\alpha}\}\rangle$

- **α-cuts uniquely identify the original constraints and networks**
  - If $\forall\alpha$ $C_{1\alpha} = C_{2\alpha}$ then $C_1 = C_2$
  - If $\forall\alpha$ $\mathcal{N}_{1\alpha} = \mathcal{N}_{2\alpha}$ then $\mathcal{N}_1 = \mathcal{N}_2$

The key property of α-cut

Theorem 2. *Given a fuzzy constraint network $\mathcal{N}$*

$$[SOL(\mathcal{N})]_{\alpha} = SOL(\mathcal{N}_{\alpha})$$

Sketch of proof.

- $\overline{d} \in [\mathrm{SOL}(\mathcal{N})]_{\alpha}$ if and only if it satisfies the worst constraint with a degree $\geq \alpha$
- this can happen if and only $\overline{d}$ satisfies *all* constraints with a degree $\geq \alpha$
- in turn, this can happen if and only if $\overline{d}$ satisfies all the α-cuts of $\mathcal{N}$, i.e. $\overline{d} \in \mathrm{SOL}(\mathcal{N}_{\alpha})$

□

From crisp to fuzzy reasoning frameworks

- **Crisp projection** of a class $\mathcal{FN}$ of fuzzy constraint networks:

$$\mathcal{C}(\mathcal{FN}) = \{\mathcal{N}_{\alpha} \mid \alpha \in [0,1], \mathcal{N} \in \mathcal{FN}\}$$

- **Fuzzy extension** of a class $\mathcal{HN}$ of crisp constraint networks:

$$\mathcal{FN} \in \mathcal{F}(\mathcal{HN}) \text{ iff } \mathcal{C}(\mathcal{FN}) \subseteq \mathcal{HN}$$

- We consider the **proper** fuzzy extension, i.e. that including *all* fuzzy networks satisfying the condition above

Examples

$\mathcal{FN}_{TCSP}$: **Fuzzy extension of TCSP**

- variables: time points
- domains: $\Re$
- constraints: binary, of the form $C_{ij} : \langle I, f \rangle$, where $f : I \to [0, 1]$ and $I = \{(a_1, b_1), \ldots (a_n, b_n)\}$

IA^{fuz}: **Fuzzy extension of Interval Algebra**

- variables: time intervals
- domains: $\Re^2$
- constraints: binary, of the kind $I_1(b\,[0.3]\,, m\,[0.5])I_2$

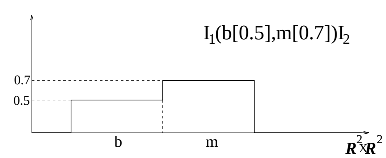
Syntax and semantics of IA^{fuz}

Syntax: IA^{fuz} is defined on the set

$$\begin{aligned} I = \{&b[\alpha_1], a[\alpha_2], m[\alpha_3], mi[\alpha_4], d[\alpha_5] di[\alpha_6], o[\alpha_7], \\ &oi[\alpha_8], s[\alpha_9], si[\alpha_{10}], f[\alpha_{11}], fi[\alpha_{12}], eq[\alpha_{13}]\} \end{aligned}$$

where $\alpha_i \in [0, 1], i = 1, \ldots, 13$

Semantics

- Atomic relation: fuzzy subset of $\Re^2 \times \Re^2$
- Generic relation: union of fuzzy subsets
- Example: 

Extending tractable classes

- Tractable class of crisp networks: there is a polynomial algorithm

$$\mathrm{SOLALG}_{\mathcal{HN}}(\mathcal{N}) = \begin{cases} \overline{d} : \overline{d} \in \mathrm{SOL}(\mathcal{N}) \text{ if } \mathrm{SOL}(\mathcal{N}) \neq \emptyset \\ \text{FAILED otherwise} \end{cases}$$

- Tractable class of fuzzy networks: there is a polynomial algorithm able to find an optimal solution (thus also to compute the consistency degree of the network)

Theorem 3. *Let $\mathcal{HN}$ be a* tractable *class of crisp networks. If $\mathcal{FN}$ is a fuzzy extension of $\mathcal{HN}$ such that $\forall \mathcal{N} \in \mathcal{FN}$ the number of preference degrees is at most exponential in the number of variables, then $\mathcal{FN}$ is tractable.*

Sketch of proof.

- Given a network $\mathcal{N} \in \mathcal{FN}$, the set of the optimal solutions is $[\mathrm{SOL}(\mathcal{N})]_\beta$, where β is the maximum α such that $[\mathrm{SOL}(\mathcal{N})]_\alpha \neq \emptyset$
- By the key property, $[\mathrm{SOL}(\mathcal{N})]_\beta = \mathrm{SOL}(\mathcal{N}_\beta)$: we can work on crisp networks
- Thus, we can perform a binary search (logarithmic complexity) on the preference degrees of $\mathcal{N}$, exploiting $\mathrm{SOLALG}_{\mathcal{HN}}$ to check consistency of $\mathrm{SOL}(\mathcal{N}_\alpha)$ for different values of α

□

Fuzzy extension of simple temporal problems

STP (Simple Temporal Problem) : $\mathcal{HN}_{STP}$

- variables: time points
- domains: $\Re$
- constraints: binary, of the form $C_{ij} : x_j - x_i \in [a_i, b_j]$
- $\mathcal{HN}_{STP}$ is tractable

$\mathcal{FN}_{STP}$: the fuzzy extension of $\mathcal{HN}_{STP}$

- variables: time points
- domains: $\Re$
- constraints: binary, of the form $C_{ij} : \langle [a_i, b_j], f\rangle$ where $f : [a_i, b_j] \to [0, 1]$ and f is semi-convex, i.e. $\forall y \ \{x \mid f(x) \geq y\}$ forms an interval
- $\mathcal{FN}_{STP}$ is tractable

Tractable subclasses of IA^{fuz}

Tractable subalgebras of classical IA

- SA_c: IA-relations that can be expressed by PA_c-relations between endpoints, i.e. PA-relations without $\neq$
- SA: IA-relations that can be expressed by PA-relations between endpoints
- $\mathcal{H}$: maximal tractable subalgebra introduced by Nebel, including so-called pre-convex relations of IA

Tractable subalgebras of IA^{fuz}

- ${SA_c}^{fuz}$ (and similarly SA^{fuz} and $\mathcal{H}^{fuz}$) can be defined as the set of relations $\{R \in IA^{fuz} \mid \forall\alpha\ R_\alpha \in SA_c\}$
- All these subalgeras are tractable
- More on this later

Fuzzy extension of properties

- Specific properties can be exploited by algorithms (e.g. path consistency entails minimality in some subclasses)
- Property of a class $\mathcal{GN}$ of crisp or fuzzy constraint networks:

$$P : \mathcal{GN} \to \{0, 1\}$$

- Given a property P defined on a crisp class $\mathcal{HN}$ and a fuzzy class $\mathcal{FN} \in \mathcal{F}(\mathcal{HN})$

$$P^{fuz}(\mathcal{N}) = \begin{cases} 1 \text{ if } \forall\alpha \in [0,1]\ \ P(\mathcal{N}_\alpha) = 1 \\ 0 \text{ otherwise} \end{cases}$$

- It can be shown that if ${P_1}^{fuz}$ and ${P_2}^{fuz}$ are the fuzzy extensions of P_1 and P_2 respectively, then $({P_1}^{fuz} \wedge {P_2}^{fuz})$ is the fuzzy extension of $(P_1 \wedge P_2)$

Fuzzy extension of important properties

Theorem 4. *Given a crisp class $\mathcal{HN}$ and a fuzzy class $\mathcal{FN} \in \mathcal{F}(\mathcal{HN})$, k-consistency on $\mathcal{FN}$ is the fuzzy extension of k-consistency on $\mathcal{HN}$*

Sketch of proof.

- By definition, we have to prove that for any $\mathcal{N} \in \mathcal{FN}$, $\mathcal{N}$ is k-consistent iff $\forall\alpha \in [0,1]\ \ \mathcal{N}_\alpha$ is k-consistent
- Assume $\mathcal{N}$ is k-consistent; consider $\alpha \in [0,1]$ and $\mathcal{N}_\alpha$:
  - any consistent instantiation $\overline{d}$ of $k-1$ variables in $\mathcal{N}_\alpha$ belongs to $\mathrm{SOL}(\mathcal{N}_\alpha^{k-1}) = (\mathrm{SOL}(\mathcal{N}^{k-1}))_\alpha$ (key prop.)
  - by k-consistency of $\mathcal{N}$, $\overline{d}$ can be extended to any additional variable maintaining the consistency degree α
  - $\overline{d}d_k \in (\mathrm{SOL}(\mathcal{N}^k))_\alpha = \mathrm{SOL}(\mathcal{N}_\alpha^k)$ (key prop.)

□

Fuzzy extension of important properties (2)

Sketch of proof (2).

- Assume $\forall\alpha \in [0,1]\ \ \mathcal{N}_\alpha$ is k-consistent; we have to prove that $\mathcal{N}$ is k-consistent:
  - any instantiation $\overline{d}$ of $k-1$ variables with $cons(\overline{d}) = \beta$ belongs to $(\mathrm{SOL}(\mathcal{N}^{k-1}))_\beta = \mathrm{SOL}(\mathcal{N}_\beta^{k-1})$ (key prop.)

– by k-consistency of $\mathcal{N}_\beta$, $\overline{d}$ can be extended to any additional variable maintaining consistency
– $\overline{d}d_k \in \mathrm{SOL}(\mathcal{N}_\beta^k) = (\mathrm{SOL}(\mathcal{N}^k))_\beta$ (key prop.)

□

Corollary 5. *Fuzzy path-consistency is the fuzzy extension of classical path-consistency.*

Fuzzy extension of important properties (3)

Theorem 6. *Fuzzy minimality is the fuzzy extension of classical minimality.*

Proof.

- We have to prove that, given $\mathcal{N} \in \mathcal{FN}$, $\mathcal{N}$ is minimal if and only if $\forall\alpha \in [0, 1]$ $\mathcal{N}_\alpha$ is minimal.
- The proof proceeds in a similar way as the one for k-consistency, exploiting the key property of α-cuts.

□

Extending theorems from crisp to fuzzy

Theorem 7. *If we have a theorem in a crisp class $\mathcal{HN}$ of the form*

$$\forall\mathcal{N} \in \mathcal{HN}\ P_1(\mathcal{N}) \Rightarrow P_2(\mathcal{N})$$

then the following theorem holds in $\mathcal{FN} \in \mathcal{F}(\mathcal{HN})$:

$$\forall\mathcal{N} \in \mathcal{FN}\ P_1^{fuz}(\mathcal{N}) \Rightarrow P_2^{fuz}(\mathcal{N})$$

Proof.

- If $P_1^{fuz}(\mathcal{N})$, then by definition $\forall\alpha \in [0, 1]$ $P_1(\mathcal{N}_\alpha)$ holds.
- By the theorem in $\mathcal{HN}$ $\forall\alpha \in [0, 1]$ $P_2(\mathcal{N}_\alpha)$ holds.
- Then $P_2^{fuz}(\mathcal{N})$ holds by definition.

□

Some direct results

- As for classical simple temporal problems, in $\mathcal{FN}_{STP}$ path-consistency entails minimality
- As for classical SA_c, in ${SA_c}^{fuz}$ path-consistency entails minimality
- As for classical SA, in SA^{fuz} minimality of 4-subnetworks entails minimality

Extending algorithms from crisp to fuzzy

- Algorithms that compute transformation of networks: given a class $\mathcal{GN}$ of fuzzy/crisp networks

$$\begin{array}{l}\mathcal{GN}\text{-T-ALG} \quad A : \mathcal{GN} \to \mathcal{GN} \\ \text{ such that } A(\langle X, D, C\rangle) = (\langle X, D, C_{out}\rangle)\end{array}$$

- $\mathcal{GN}$-T-ALG equivalence preserving conditioned on P (P-EQ)

$$\forall \mathcal{N} \in \mathcal{GN} \;\; P(\mathcal{N}) \to \mathrm{SOL}(A(\mathcal{N})) = \mathrm{SOL}(\mathcal{N})$$

- $\mathcal{GN}$-T-ALG enforcing P_2 conditioned on P_1 (P_1-to-P_2)

$$\forall \mathcal{N} \in \mathcal{GN} \;\; P_1(\mathcal{N}) \to P_2(A(\mathcal{N}))$$

Extending algorithms from crisp to fuzzy (2)

- Fuzzy extension of a $\mathcal{HN}$-T-ALG A to $\mathcal{FN}$, where $\mathcal{FN} \in \mathcal{F}(\mathcal{HN})$:

$$\begin{array}{l}\mathcal{FN}\text{-T-ALG} \quad A^{fuz} \text{ such that} \\ \forall \mathcal{N} \in \mathcal{FN}, \forall \alpha \in [0,1]\,, \;\; (A^{fuz}(\mathcal{N}))_\alpha = A(\mathcal{N}_\alpha)\end{array}$$

- Results:
  - A^{fuz} is guaranteed to exist provided any network has a finite number of preference degrees
  - If A is P-EQ, then A^{fuz} is P^{fuz}-EQ
  - If A is P_1-to-P_2, then A^{fuz} is P_1^{fuz}-to-P_2^{fuz}

Conclusions

- The methodology also holds using other operators besides *min*, provided idempotency holds
- Main message: some classical results can be directly extended to a fuzzy framework

Part III
Fuzzy qualitative temporal reasoning

Fuzzy qualitative temporal reasoning

Contents

4 The algebra IA^{fuz}

Dutta's and Guesgen's approaches

Dutta's approach

- A set of precise and disjoint intervals assumed as background
- Initial representation about events: $\mu_i(e) \equiv$ degree of possibility that interval i contains event e
- Infer the possibility degree that a relation in $\{b, a, m\}$ holds between two events

Guesgen et al.

- Focus in *imprecise spatial descriptions*
- Imprecision of observations expressed by fuzzy values associated to Allen's atomic relations

Both approaches can be expressed by a fragment of IA^{fuz}

Syntax and semantics of IA^{fuz}

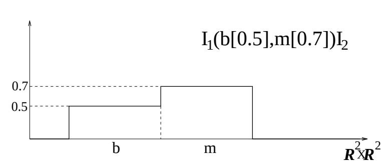
Syntax: IA^{fuz} is defined on the set

$$I = \{b[\alpha_1], a[\alpha_2], m[\alpha_3], mi[\alpha_4], d[\alpha_5] di[\alpha_6], o[\alpha_7], oi[\alpha_8], s[\alpha_9], si[\alpha_{10}], f[\alpha_{11}], fi[\alpha_{12}], eq[\alpha_{13}]\}$$

where $\alpha_i \in [0, 1], i = 1, \ldots, 13$

Semantics

- Atomic relation: fuzzy subset of $\Re^2 \times \Re^2$
- Generic relation: union of fuzzy subsets



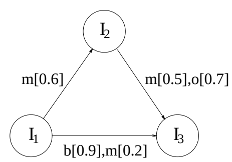
Intended meaning Preference between IA-relations, e.g. A_1 should be disjoint w.r.t. A_2, and it's better A_1 before A_2

Local consistency in IA^{fuz} networks

- Singleton labeling (assignment): choice of an atomic relation for every pair of intervals
- Degree of local consistency:

$$deg_{\mathcal{N}}(s) = \begin{cases} 0 & \text{if } s \text{ is not consistent} \\ \min_{(i,j)} R_{ij}(s_{ij}) & \text{otherwise} \end{cases}$$

Example:



$(I_1\ m\ I_2, I_2\ m\ I_3, I_1\ b\ I_3) : 0.5$ $(I_1\ m\ I_2, I_2\ m\ I_3, I_1\ m\ I_3) : 0$

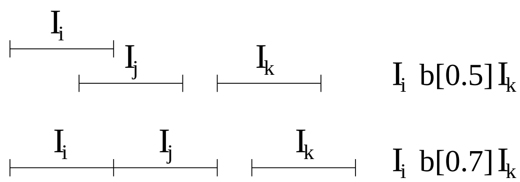
Operations of the algebra IA^{fuz}

- Inversion
$$R^{-1} = ({rel_1}^{-1}[\alpha_1], \ldots, {rel_{13}}^{-1}[\alpha_{13}])$$
- Conjunctive combination $R = R' \otimes R''$
$$R = (rel_1[\alpha_1], \ldots, rel_{13}[\alpha_{13}])$$
$$\alpha_i = \min\{\alpha'_i, \alpha''_i\} \quad i \in \{1, \ldots, 13\}$$
- Disjunctive combination $R = R' \oplus R''$
$$R = (rel_1[\alpha_1], \ldots, rel_{13}[\alpha_{13}])$$
$$\alpha_i = \max\{\alpha'_i, \alpha''_i\} \quad i \in \{1, \ldots, 13\}$$

Operations of the algebra IA^{fuz} (2)
Composition

- Atomic relations:
$$rel_1[\alpha_1] \circ rel_2[\alpha_2] = (rel'_1[\alpha], rel'_2[\alpha], \ldots, rel'_l[\alpha])$$
$$\text{where} \quad rel'_i \in \{rel_1 \circ rel_2\} \text{ and } \alpha = \min\{\alpha_1, \alpha_2\}$$
- Generic relations: by distributivity property
$$R' \circ R'' = (rel_1[\alpha_1], \ldots, rel_{13}[\alpha_{13}])$$
$$\alpha_p = \max_{q,r:rel_p \in \{rel_q \circ rel_r\}} \min\{\alpha'_q, \alpha''_r\}$$
$$p, q, r \in \{1, \ldots, 13\}$$
- Intuitively: α_p is the degree through which rel_p can be extended to a labeling involving R' and R''

Example of composition



$R_{ij} = (o[0.5], m[0.7])$ $\quad R_{jk} = (b[0.9])$

$$R_{ij} \circ R_{jk} = (o[0.5], m[0.7]) \circ (b[0.9]) = (b[0.5] \oplus b[0.7]) = b[0.7]$$

Interesting reasoning tasks in IA^{fuz}

- Determining the consistency degree of an IA^{fuz}-network
- Finding an optimal solution (i.e. singleton labeling)
- Computing the minimal network
- Equivalent under polynomial Turing-reduction

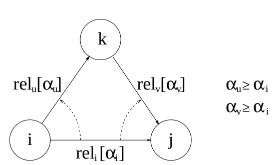
Algorithms

- Constraint propagation algorithms: mainly related to minimality, e.g.
  - PC^{fuz}: enforces path-consistency
  - AAC^{fuz}: enforces minimality of 4-subnetworks

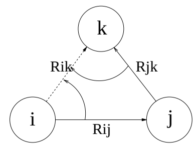
  Extend classical algorithms, but with specific improvements
- Branch & Bound algorithm: computes an optimal solution

Path-consistency algorithm

- Path consistency enforced if and only if $\forall(i, j, k)\ R_{ij} \leq (R_{ik} \circ R_{kj})$



- Basic idea: applying transitivity rules



- Since IA^{fuz} operations generalize the classical ones, the classical PC-algorithm is still valid.

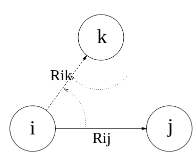
The original path-consistency algorithm

$PC^{fuz}(\mathcal{N})$

$Q \leftarrow \{(i,j) \mid 1 \leq i < j \leq n\}$
while $(Q \neq \emptyset)$
    `select and delete` (i,j) `from` Q
    for $k \leftarrow 1$ `to` n, $k \neq i$ `and` $k \neq j$
        $t \leftarrow R_{ik} \otimes (R_{ij} \circ R_{jk})$
        if $(t \neq R_{ik})$
        then $R_{ik} \leftarrow t$
            $R_{ki} \leftarrow t^{-1}$
            $Q \leftarrow Q \cup \{(i,k)\}$
        $t \leftarrow R_{kj} \otimes (R_{ki} \circ R_{ij})$
        if $(t \neq R_{kj})$
        then $R_{kj} \leftarrow t$
            $R_{jk} \leftarrow t^{-1}$
            $Q \leftarrow Q \cup \{(k,j)\}$

Improvements

- Not labeled edges



$$R_{ij} \circ R_{jk} = I[\alpha^*_{ij}] \ \alpha^*_{ij} = \max\{\alpha^{ij}_1, \ldots, \alpha^{ij}_{13}\}$$

- Con-Sup $= \min_{(i,j)} \{\alpha^*{}_{ij}\}$
  - When Con-Sup decreases $\forall (i,j)\ R_{ij} \leftarrow R_{ij} \otimes I[\text{Con-Sup}]$
  - However, it's the same to apply this truncation to edges involved in $R_{ik} \leftarrow R_{ik} \otimes (R_{ij} \circ R_{jk})$ + final truncation
- $R_{ik} \leftarrow R_{ik} \otimes (R_{ij} \circ R_{jk})$ only if $\min^*_{ij} <$ Con-Sup and $\min^*_{jk} <$ Con-Sup
- Insert an edge into Q only if a preference degree strictly lower than Con-Sup has been modified

The improved path-consistency algorithm

```
PC2^fuz(N)
Q ← {(i, j) | 1 ≤ i < j ≤ n, min_ij < ConsSup}
while (Q ≠ ∅)
        select and delete (i, j) from Q
        if (min_ij < ConsSup)
        then for k ← 1 to n, k ≠ i and k ≠ j
                    if (min_jk < ConsSup)
                    then t ← R_ik ⊗ (R_ij ∘ R_jk)
                         if (∃rel_p : deg_t(rel_p) < min {ConsSup, deg_R_ik(rel_p)})
                         then R_ik ← t
                              R_ki ← t^-1
                              Q ← Q ∪ {(i, k)}
                              ConsSup = min {ConsSup, max_ik}
                    if (min_ki < ConsSup)
                    then ...
...
∀(i, j) R_ij ← R_ij ⊗ I[ConsSup]
return ConsSup
```

Branch & Bound Algorithm

1. Application of $PC2^{fuz}$ Algorithm; $\alpha_{\inf} = 0$, $\alpha_{\sup} =$ Con-Sup.
2. If Con-Sup > 0, consider every edge in a fixed order.
3. For the current (i, j): choose $\beta_{ij} \mid \mathrm{pref}(\beta_{ij}) > \alpha_{\inf}$; $R_{ij} \leftarrow \beta_{ij}[\mathrm{pref}(\beta_{ij})]$; P.C. Algorithm.
4. If Con-Sup $\leq \alpha_{\inf}$ then choose another β_{ij} or backtrack to the precedent edge.
5. Complete assignment: If Con-Sup $> \alpha_{\inf}$, best current solution, $\alpha_{\inf} \leftarrow$ Con-Sup, test $\alpha_{\inf} = \alpha_{\sup}$.

Pointizable algebras: SA^{fuz} and SA_c^{fuz}

Fuzzy extensions of classical PA and PA_c

- PA^{fuz} algebra: relations between points of the form $\{< [\alpha_1], = [\alpha_2], > [\alpha_3]\}$
- PA_c^{fuz} algebra: PA^{fuz} relations with $\alpha_2 \geq \min\{\alpha_1, \alpha_3\}$

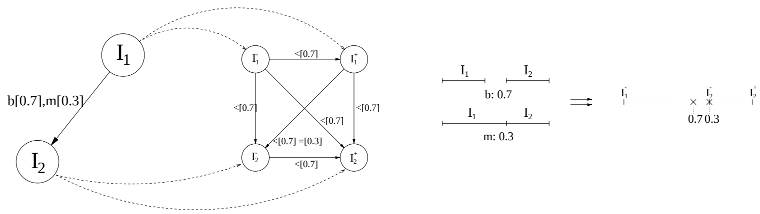
Fuzzy extensions of classical SA and SA_c

- SA^{fuz}: IA^{fuz} relations that can be expressed as PA^{fuz} relations between endpoints
- SA_c^{fuz}: relations that can be expressed as PA_c^{fuz} relations

All of these sets are algebras (can be proved by exploiting the relationships between classical and fuzzy operations by means of α-cuts).

Example of SA_c^{fuz} relation

- The IA^{fuz} relation $(b[0.7], m[0.3])$ can be translated into the following PA-network



- Since point relations belong to PA_c^{fuz}, $(b[0.7], m[0.3]) \in SA_c^{fuz}$

Tractability of SA^{fuz} and SA_c^{fuz}

- Main properties:
$$R \in SA^{fuz} \text{ iff } \forall \alpha \ R_\alpha \in SA$$
and
$$R \in SA_c^{fuz} \text{ iff } \forall \alpha \ R_\alpha \in SA_c$$

- SA_c^{fuz}: path-consistency entails minimality, thus the minimal network can be computed in $O(kn^3)$

- SA^{fuz}: minimality of 4-subnetworks entails minimality, thus the minimal network can be computed in $O(kn^4)$

A maximal tractable subalgebra of IA^{fuz}

- Nebel's $\mathcal{H} \subseteq IA$ is a maximal tractable algebra:
  - path-consistency entails $\mathcal{N}$ consistent iff $\forall i, j \ R_{ij} \neq \emptyset$. Thus, consistency can be checked in $O(n^3)$
  - if $\mathcal{N}$ is path-consistent, a solution can be computed without backtrack in $O(n^2)$ (Ligozat, 98)

- Definition: $R \in \mathcal{H}^{fuz}$ iff $\forall \alpha \ R_\alpha \in \mathcal{H}$

- Properties:
  - If $\mathcal{N}$ is path-consistent, max_{ij} for any edge ij gives the consistency degree, Ligozat's algorithm applied to $\mathcal{N}_{max_{ij}}$ gives an optimal solution
  - $\mathcal{H}^{fuz}$ is the unique maximal tractable subalgebra of IA^{fuz} which includes all the relations of $\mathcal{B} = \{rel_p[\alpha] \mid rel_p \in IA, \alpha \in [0,1]\}$

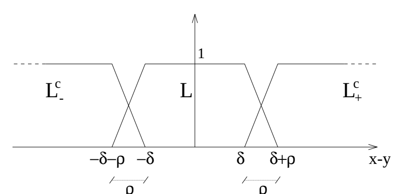
5 Dubois, HadjAli & Prade approach

Modeling fuzzy Allen relations

Motivation The relations holding between intervals may not be described in precise terms: need to express relations of the kind "approximately equal", "much before" etc. in order to avoid brutal discontinuities. **Basis of the modeling**

- Definition of the fuzzy counterparts of classical relations between points:
  - $<$ becomes "much smaller"
  - $=$ becomes "approximately equal"
  - $>$ becomes "much greater"
- Definition of the fuzzy counterparts of classical Allen relations on the basis of fuzzy relations between their endpoints

Modeling approximate equality and graded inequalities between points



$\forall d\ \mu_{L^c_-}(d) + \mu_L(d) + \mu_{L^c_+}(d) = 1$

- Fuzzy counterparts of classical relations:
  - $a < b$ replaced by $a\ S(L^c_-)\ b$
  - $a = b$ replaced by $a\ E(L)\ b$
  - $a > b$ replaced by $a\ G(L^c_+)\ b$
- Parameters: δ and ρ (if $\delta = 0$ and $\rho \to 0$ classical relations are recovered)

Fuzzy Allen relations

where $a = [a, a']$, $b = [b, b']$

Composition of fuzzy relations between points

Composition of relations $G(K)$ and $E(L)$ between points:

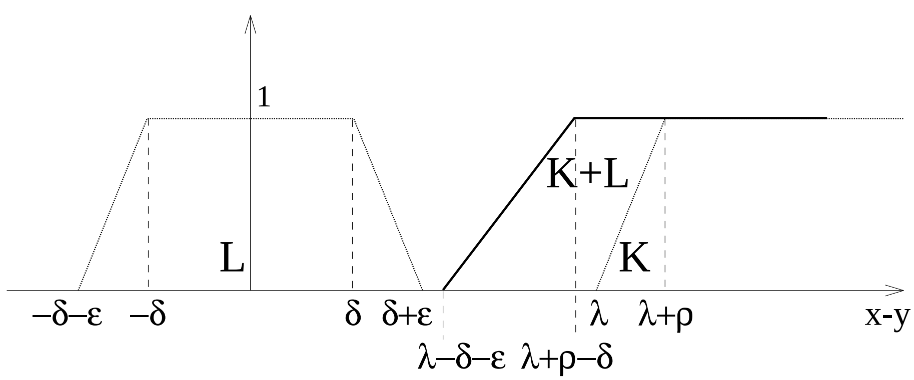
$$\begin{aligned} \forall x, z\ \mu_{G(K)\circ E(L)}(x, z) &= \sup_y \min\{\mu_G(x, y), \mu_E(y, z)\} \\ &= \mu_{K\oplus L}(x - z) \end{aligned}$$

where $\mu_{K\oplus L}(x) \equiv \sup_{s,t:x=s+t} \min\{\mu_K(s), \mu_L(t)\}$ **Example** If a is approxi-

Fuzzy Allen relation	Label	Definition
A *fuzz-before*(L) B	$fb(L)$	$b\ G(L^c_+)\ a'$
A *fuzz-meets*(L) B	$fm(L)$	$a'\ E(L)\ b$
A *fuzz-overlaps*(L) B	$fo(L)$	$b\ G(L^c_+)\ a \wedge a'\ G(L^c_+)\ b \wedge b'\ G(L^c_+)\ a'$
A *fuzz-during*(L) B	$fd(L)$	$a\ G(L^c_+)\ b \wedge b'\ G(L^c_+)\ a'$
A *fuzz-starts*(L) B	$fs(L)$	$a\ E(L)\ b \wedge b'\ G(L^c_+)\ a'$
A *fuzz-finishes*(L) B	$ff(L)$	$a'\ E(L)\ b' \wedge a\ G(L^c_+)\ b$
A *fuzz-equals*(L) B	$fe(L)$	$a\ E(L)\ b \wedge b'\ E(L)\ a'$

mately equal to b and b is much greater than c then a is much greater than c:

$$a\ E(L)\ b \wedge b\ G(K)\ c \Rightarrow \\ a\ G(K \oplus L)\ c$$



Reasoning with fuzzy Allen relations

- By composition, inference rules between points, e.g.

$$\begin{aligned} &a\ E(L)\ b\ \wedge\ b\ G(K)\ c\ \Rightarrow\ a\ G(K \oplus L)\ c \\ &a\ G(K)\ b\ \wedge\ b\ G(K')\ c\ \Rightarrow\ a\ G(K \oplus K')\ c \\ &a\ E(L)\ b\ \Rightarrow\ a + c\ E(L)\ b + c \end{aligned}$$

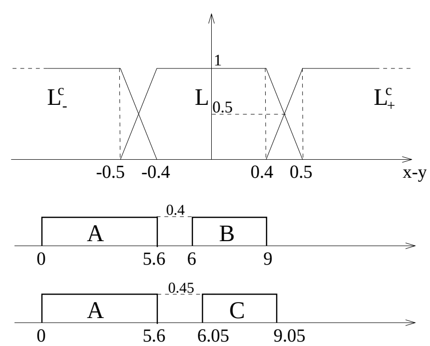
- By these rules (and the fact that fuzzy Allen relations can be expressed as rules between endpoints), transitivity rules between fuzzy Allen relations, e.g.
$$A\ fb(L_1)\ B \wedge B\ fb(L_2)\ C \Rightarrow A\ fb(L_2 \oplus L_1)\ C$$

- A 13×13 composition table is defined.

Complete vs. uncertain information

- Available temporal information (i.e. about time points and relative positions of intervals) is *complete*, but we are interested in *evaluating fuzzy statements* (i.e. approximate equality or proximity) in order to avoid discontinuities.
- Available temporal information is *imprecise, vague or uncertain*, and we are interested in evaluating crisp or fuzzy statements.

Example: evaluation of fuzzy Allen relations between crisp intervals

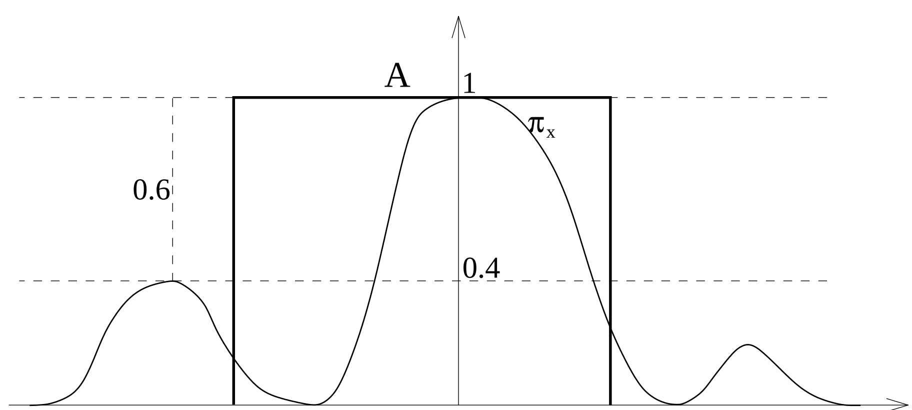


- $A\ fb(L)\ B$ satisfied with degree 1 (since $b - a' = 0.4$)
- $A\ fb(L)\ C$ and $A\ fm(L)\ C$ satisfied with degree 0.5 (since $b - a' = 0.45$)

Background: possibility and necessity measures

Given a variable x with associated possibility distribution $\pi(x)$ and a *crisp* set A:

- Possibility of $x \in A$: $\Pi(A, x) = \sup_{x \in A} \pi(x)$
- Necessity of $x \in A$: $N(A, x) = 1 - \Pi(\overline{A}, x) = \inf_{x \notin A} \pi(x)$



$\Pi(A, x) = 1, N(A, x) = 0.6$ If A is a fuzzy set:

- Possibility of x *is* A: $\Pi(A, x) = \sup_x \min\{\mu_A(x), \pi(x)\}$
- Necessity of x *is* A: $N(A, x) = 1 - \Pi(\overline{A}, x) = \inf_x \max\{\mu_A(x), 1 - \pi(x)\}$

Possibility and necessity measures: basic properties

It is easy to verify that, for all A and B:

- $\Pi(A \cup B, x) = \max\{\Pi(A, x), \Pi(B, x)\}$
- $N(A \cap B, x) = \min\{N(A, x), N(B, x)\}$

while it holds that

- $\Pi(A \cap B, x) \leq \min\{\Pi(A, x), \Pi(B, x)\}$
- $N(A \cup B, x) \geq \max\{N(A, x), N(B, x)\}$

Uncertain relations between points

Given information about the possible location of dates a and b expressed by π_a and π_b respectively, it turns out that:

- $N(a > b) = 1 - \sup_{s \leq t} \min\{\pi_a(s), \pi_b(t)\}$
- $N(a\ G(K)\ b) = \inf_{s,t} \max\{\mu_G(s,t), 1 - \pi_a(s), 1 - \pi_b(t)\}$
- $N(a\ E(L)\ b) = \inf_{s,t} \max\{\mu_L(s,t), 1 - \pi_a(s), 1 - \pi_b(t)\}$

For instance, by the formula of the necessity of x *is* A

$$\begin{aligned} N(a\ G(K)\ b) &= \inf_{s,t} \max\{\mu_G(s,t), 1 - \pi_{(a,b)}(s,t)\} \\ &= \inf_{s,t} \max\{\mu_G(s,t), 1 - \min(\pi_a(s), \pi_b(t))\} \\ &= \inf_{s,t} \max\{\mu_G(s,t), 1 - \pi_a(s), 1 - \pi_b(t)\} \end{aligned}$$

Certainty degrees of Allen relations

Recalling that $N(A \cap B, x) = \min\{N(A, x), N(B, x)\}$, the necessity degrees of ordinary Allen relations can be expressed w.r.t. the necessity of endpoints relations, e.g.:

- $N(a\ \textit{before}\ b) = N(b > a')$
- $N(a\ \textit{overlaps}\ b) = \min\{N(b > a), N(a' > b), N(b' > a')\}$

Similarly for fuzzy Allen relations, e.g.

- $N(A\ fb(L)\ B) = N(b\ G(L_+^c)\ a')$
- $N(A\ fo(L)\ B) = \min\{N(b\ G(L_+^c)\ a), N(a'\ G(L_+^c)\ b), N(b'\ G(L_+^c)\ a')\}$

$$\frac{\begin{array}{l} N(A\ fm(L_1)\ B) \geq \alpha \\ N(C\ fs(L_2)\ B) \geq \beta \end{array}}{N(C\ fm(L_1 \oplus L_2)\ A) \geq \min\{\alpha, \beta\}}$$

Patterns of inference with fuzzy Allen relations

By transitivity rules of $N()$ and the above definitions, several reasoning patterns can be derived, e.g. This way, it is possible to handle and reason with statements of the kind "It is certain to the degree α that A fuzzily meets B"

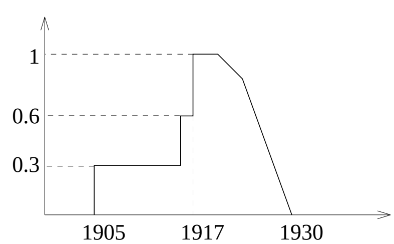
6 Nagypál and Motik approach

Context and motivation

- Modeling information about historical events: *uncertain* (e.g. contradictory documents), *subjective* (unclear definitions, e.g. "the industrial revolution") and *vague*
- Main requirement: given a number of possibly imprecise temporal specifications using absolute dates (events), deduce Allen relations between events [no general reasoning capability required]
- When applied to traditional (i.e. non vague) specifications, the same results as in classical temporal models should be obtained

Time intervals as fuzzy sets

- An event i is modeled as a time interval corresponding to a fuzzy set $\widetilde{I}$, where $\mu_{\widetilde{I}}(t)$ expresses the *confidence level* that t is in i (due to uncertainty, subjectivity and vagueness)
- Example: Russian Revolution

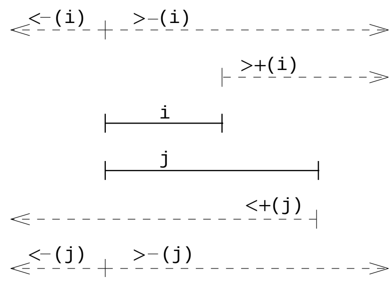


Fuzzy temporal relations

- Meaning: given two events i and j (modeled by the fuzzy sets $\widetilde{I}$ and $\widetilde{J}$ respectively) and a fuzzy temporal relation $\widetilde{\theta}$ corresponding to a crisp Allen relation θ, $\widetilde{\theta}$ takes $\widetilde{I}$ and $\widetilde{J}$ and produces a number $c \in [0,1]$ expressing the confidence that θ holds between i and j

- Definition in two steps:
  - express classical Allen relations without reference to endpoints (claimed to be meaningless with fuzzy intervals)
  - fuzzify the obtained relations

First step



Consider e.g. i *starts* j

$$i \textit{ starts } j \;\equiv\; \begin{array}{l} >-(i) \cap \;<-(j) = \emptyset \,\wedge \\ >-(j) \cap \;<-(i) = \emptyset \,\wedge \\ >+(i) \cap \;<+(j) \neq \emptyset \end{array}$$

Auxiliary operators ($<-$, $\leq -$, $>-$, $\geq -$, $<+$, $\leq +$, $>+$, $\geq +$) are defined

Second step

$$i \textit{ starts } j \;\equiv\; \begin{array}{l} >-(i) \cap \;<-(j) = \emptyset \,\wedge \\ >-(j) \cap \;<-(i) = \emptyset \,\wedge \\ >+(i) \cap \;<+(j) \neq \emptyset \end{array} \qquad \begin{array}{l} STARTS(\widetilde{I}, \widetilde{J}) \;\equiv\; \min\{ \\ \inf_t \max\{\widetilde{I}_{\leq -}(t), \widetilde{J}_{\geq -}(t)\}, \\ \inf_t \max\{\widetilde{I}_{\geq -}(t), \widetilde{J}_{\leq -}(t)\}, \\ \sup_t \min\{\widetilde{I}_{>+}(t), \widetilde{J}_{<+}(t)\}\} \end{array} \quad \text{where}$$

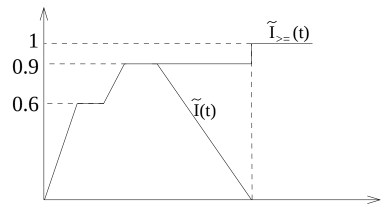
- the confidence that $a \cap b \neq \emptyset$ is $\sup_t \min\{\widetilde{A}(t), \widetilde{B}(t)\}$
- the confidence that $a \cap b = \emptyset$ is $1 - \sup_t \min\{\widetilde{A}(t), \widetilde{B}(t)\} = \inf_t \max\{\widetilde{A}^c(t), \widetilde{B}^c(t)\}$

Extending auxiliary operators

- Meaning of $\widetilde{\theta}$ extending θ: $\widetilde{\theta}(\widetilde{I})(t)$ gives the confidence that t is in $\theta(i)$
- The operator $\geq - : \widetilde{I} \to \widetilde{I}$

$$\widetilde{I}_{\geq -}(t) = \begin{cases} 0 \text{ if } t < S^{-}_{\widetilde{I}} \\ \sup_{s \leq t} \widetilde{I}(s) \text{ if } t \in S_{\widetilde{I}} \\ 1 \text{ if } t > S^{+}_{\widetilde{I}} \end{cases}$$

- Example:



7 Ohlbach's approach

Motivation

- Similarly to Nagypál & Motik approach, represent fuzzy time intervals
- Differently from Nagypál & Motik (but similarly to Dubois et al.), represent fuzzy relations even in case of crisp intervals (e.g. consider the DB query "give me all performances ending before midnight")
- Similarly to Nagypál & Motik, no general reasoning: from known (possibly fuzzy) time intervals to fuzzy relations between them
- Customizable relations (operator-based)

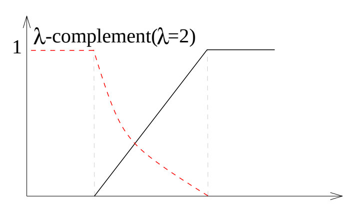
Background: general operations on fuzzy sets

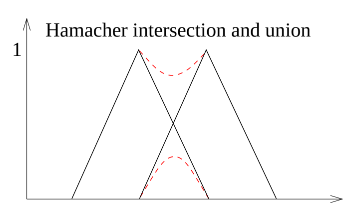
Complement n:

- $n(0) = 1$ and $n(1) = 0$
- n is non-increasing

Triangular norm T and conorm S:

- commutative, associative and monotone
- $\forall x\ T(x, 1) = x$ and $S(x, 0) = x$





Point-Interval relations

- Given a point and a (possibly fuzzy) interval, return a value $c \in [0, 1]$
- Definitions parametric w.r.t. operations on fuzzy sets
- Example: $\mathit{before}_{N,E^+}(i)$ with i finite

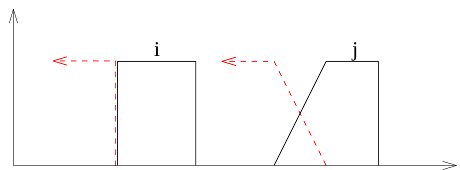
  $$\mathit{before}_{N,E^+}(i) = N(E^+(i))$$

  where N is a complement function and E^+ is a rising operator, i.e. returns an interval such that

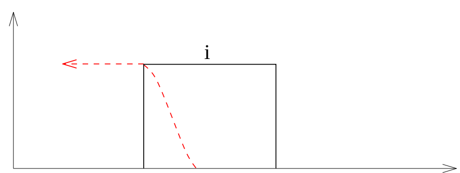
  $$E^+(i) = 1 \text{ for all } t > i^{fm}$$

before_{N,E^+}: examples

- before with standard negation and $E^+ = \mathit{extend}^+$



- a more fuzzy before exploiting a gaussian operator



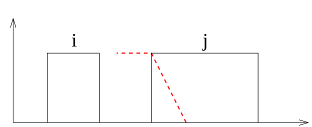
Interval-Interval relations

- Requirements: work for fuzzy time intervals, give a fuzzy value even for crisp intervals, operator-based
- Main idea: integrate a point-interval relation over the interval's membership function
- Before relation:

  $$\mathit{before}_B(i, j) = \frac{\int i(x) \cdot B(j)(x)dx}{|i|}$$

  (additional complications for non-finite intervals)

- Example (with B as in previous slide):



8 Schockaert, De Cock & Kerre approach

The basic idea

- Major aim: reasoning with fuzzy time intervals
- Reasoning concerns endpoints
- Yet a different family of fuzzy relations, e.g. from

  $$(\exists x)(x \in \left[a^-, a^+\right] \wedge (\forall y)(y \in \left[b^-, b^+\right] \Rightarrow x < y))$$

  to

  $$bb^{<<}(A, B) \equiv \sup_x T_w(A(x), \inf_y I_w(B(y), L^{<<}(x, y)))$$

- Similar definitions for $ee^{<<}$, $be^{<<}$, $eb^{<<}$, $bb^{\preceq}$, $ee^{\preceq}$, $be^{\preceq}$, and $eb^{\preceq}$
- Reduce to classical relations with crisp intervals

The reasoning task

Given a set of formulas of the kind

$$\begin{array}{l} bb^{<<}(X_1, X_2) \geq \alpha \ \vee \ be^{<<}(X_3, X_4) \geq \beta \\ eb^{<<}(X_1, X_2) \geq \gamma \ \vee \ \ldots \\ \ldots \end{array}$$

- decide satisfiability (i.e. $\exists$ an assignment of fuzzy intervals to X_i satisfying all the constraints)
- checking entailment

Main result

- A maximal tractable class of formulas where satisfiability and entailment can be checked in polynomial time
- The proof exploits the restriction that values belong to a finite set (reduction to classical point algebra)
- Involved reasoning is substantially different from e.g. constraint propagation

Overall view: a tentative classification

<table>
<tr><td></td><td colspan="5">Modeling of intervals</td></tr>
<tr><td></td><td colspan="2">Couples of points</td><td colspan="3">Vague events (fuzzy sets)</td></tr>
<tr><td>Relations</td><td>Non fuzzy</td><td>Fuzzy</td><td colspan="2">Non fuzzy</td><td>Fuzzy</td></tr>
<tr><td>Approaches</td><td>IA^{fuz}</td><td>Dubois et al.</td><td>Nagypál et al.</td><td>Schockaert et al.</td><td>Ohlbach</td></tr>
<tr><td>Reasoning</td><td>As in IA</td><td>Compos. table</td><td>Not considered</td><td>Special kind</td><td>Not considered</td></tr>
</table>

Conclusions

- A number of approaches based on different ideas and definitions
- Links between each other not yet formally investigated
- Difficult to say whether one definition is better than the other
- Mainly depends on the considered application context (e.g. scheduling, annotations of historical events, DB queries, ...)

Part IV

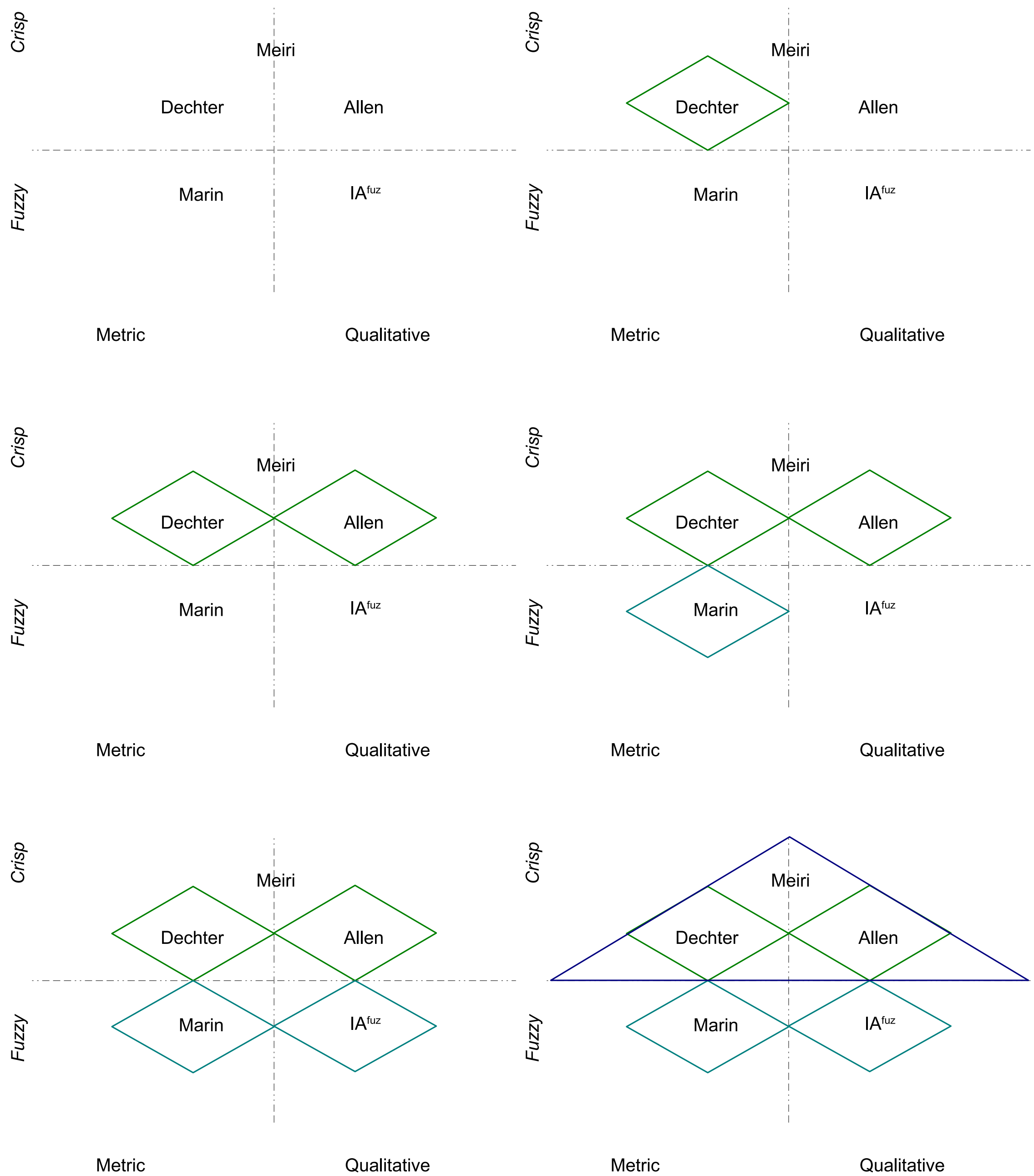
Reasoning with Qualitative and Metric Temporal Information

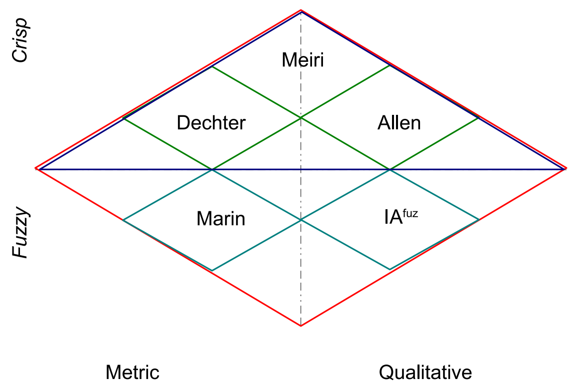
Reasoning with Fuzzy Qualitative and Metric Temporal Information

Contents

9 Background

The overall view


Crisp
Fuzzy
Meiri
Dechter
Allen
Marin
IA^{fuz}
Metric
Qualitative



9.1 Extension of QA^{fuz}

Fuzzy Point Interval Set (PI^{fuz})

Classical Point Interval relations can be extended by adding preference degrees in analogy with PA^{fuz}

A Fuzzy Point Interval relation can be written as

$$(b[\alpha_1], a[\alpha_2], d[\alpha_3], s[\alpha_4], f[\alpha_5])$$

where $\alpha_i \in [0, 1], i = 1, \ldots, 5$ are the preference degrees

Fuzzy Qualitative Algebra QA^{fuz}

The Fuzzy Qualitative Algebra between points and intervals is given by the union of:

- IA^{fuz}
- PA^{fuz}
- Fuzzy PI relations

Operations in QA^{fuz}

- Inversion and intersection operations of a relation R^{fuz} rely on the operations of the belonging algebras or sets (in the case of Fuzzy PI)

- In composition operation preference degrees are computed as in IA^{fuz}:

  $$\alpha_k = max_{u,v:rel_k \in (rel_u \circ rel_v)} min\{\alpha'_u, \alpha'_v\}$$

  where $rel_u \circ rel_v$ are the classical operations defined according to QA composition tables (see Table 1 on Slide 7)

How many relations has a fuzzy qualitative algebra?

For a given algebra with n elements there are

$$nr = n! \sum_{j=0}^{n-1} \sum_{i=1}^{|P(j)|} \frac{1}{\varphi(P_i(j))} \chi(P_i(j)) \mu(P_i(j))$$

unique full relations to be checked for tractability and

$$\chi(P_i(j)) = C^{n-j}_{|P_i(j)|},$$

$\mu(P_i(j))$ is the multinomial of $|P_i(j)|$ elements in $j-1$ groups of

$$\zeta_{P_i(j)}(k) = |\{x_h : x_h = P_{ih}(j) \wedge P_{ih}(j) = k, h = 1 \ldots |P_i(j)|\}|$$

elements and

$$\varphi(P_i(j)) = \prod_{k=1}^{|P_i(j)|} (P_{ik}(j) + 1)!$$

counts the equivalent relations

Examples

Table 3: Cardinality of fuzzy full algebras

alg.	classic rel.	fuzzy rel.
PA^{fuz}	3	13
PI^{fuz}	5	541
IA^{fuz}	13	526 858 348 381

A relation in QA^{fuz} belongs to PA^{fuz}, PI^{fuz}, $(PI^{fuz})^{-1}$ or IA^{fuz}, therefore QA^{fuz} has 526 858 349 476 relations

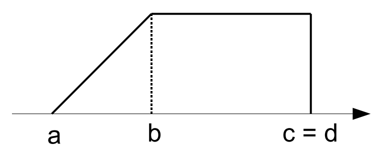
9.2 Fuzzy Metric Constraints

Fuzzy Metric constraints

Fuzzy Metric constraints can be extended to deal with preferences by associating them a possibility distribution to model preference degrees

The possibility distributions adopted are trapezoidal:

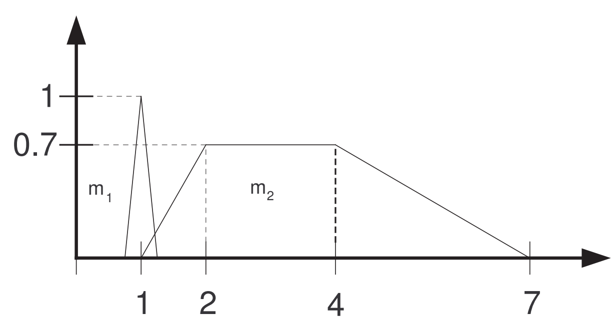
$$\lhd a, b, c, d \rhd [\alpha]$$



$a, b \in \mathbb{R} \cup \{-\infty\}$, $c_k, d_k \in \mathbb{R} \cup \{+\infty\}$
$\alpha_k \in (0, 1]$
$\lhd$ is either (or [, $\rhd$ is either) or]

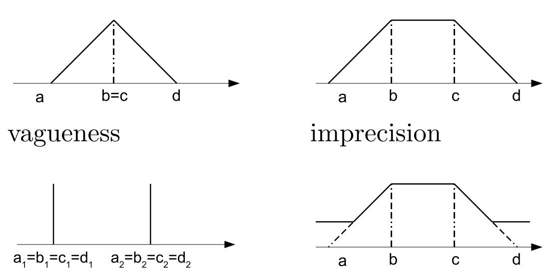
The expressiveness of trapezoidal distributions

"In disease d_1 the symptom m_1 occurs always after about a day. The symptom m_2 follows m_1 rather commonly; it uses to last between 2 to 4 days, though other less possible cases range from 1 day as the lowest bound to a week as the top one."

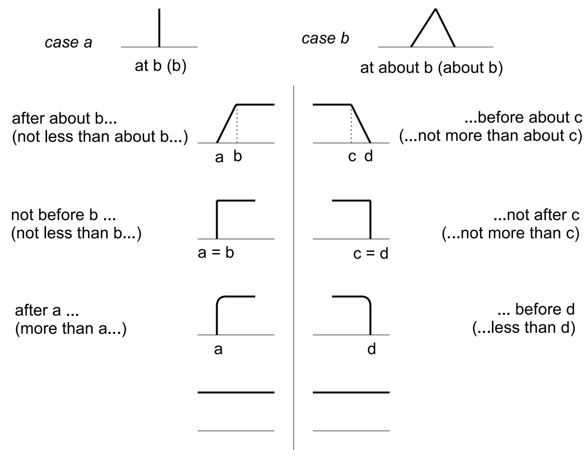


Modelling imperfect data

Fuzzy constraints can express many kinds of imperfection:



Correspondence with Natural Language expressions

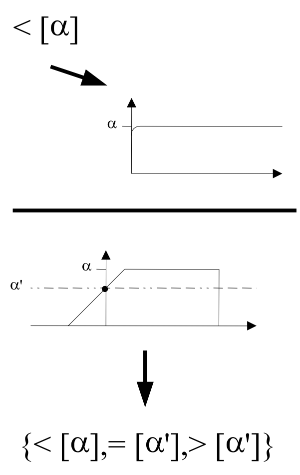


9.3 Transformation functions

Intuitions behind the transformation functions

- A qualitative relation is mapped on a semi-axis (or a point)

- A trapezoid (metric) that lies across the y axis is partitioned in three regions and then mapped on (at most) three qualitative relations



Definition of $QUAN^{fuz}$

$QUAN^{fuz}$ function transforms a qualitative fuzzy relation into a fuzzy metric constraint

Only point-point relations can be transformed

$$\begin{cases} (0,0,+\infty,+\infty)[\alpha] & \text{if } <[\alpha] \in R \\ (0,0,0,0)[\alpha] & \text{if } =[\alpha] \in R \\ (-\infty,-\infty,0,0)[\alpha] & \text{if } >[\alpha] \in R \end{cases}$$

Definition of $QUAL^{fuz}$

$QUAL^{fuz}$ function transforms a fuzzy metric constraint into a qualitative point-point fuzzy relation

$$QUAL^{fuz} = \bigcup_{k=\{<,=,>\}} QUAL_k^{fuz}$$

where

$$QUAL_<^{fuz}(R) = <[max_{i=1,\ldots,n} h_i^+]$$

$$QUAL_=^{fuz}(R) === [max_{i=1,\ldots,n} h_i^0]$$

$$QUAL_>^{fuz}(R) => [max_{i=1,\ldots,n} h_i^-]$$

Operations between mixed constraints

Let C' be metric and C" be qualitative:

- disjunction
$$C' \cup C'' = C' \cup QUAN^{fuz}(C'')$$
- conjunction
$$C' \cap C'' = C' \cap QUAN^{fuz}(C'')$$
- composition ($C'' \in PP$)
$$C' \circ C'' = C' \circ QUAN^{fuz}(C'')$$
- qualitative composition ($C'' \in PI$)
$$C' \circ C'' = QUAL^{fuz}(C') \circ C'$$

10 Tractable problems

10.1 Fuzzy metric constraints

Dealing with complexity

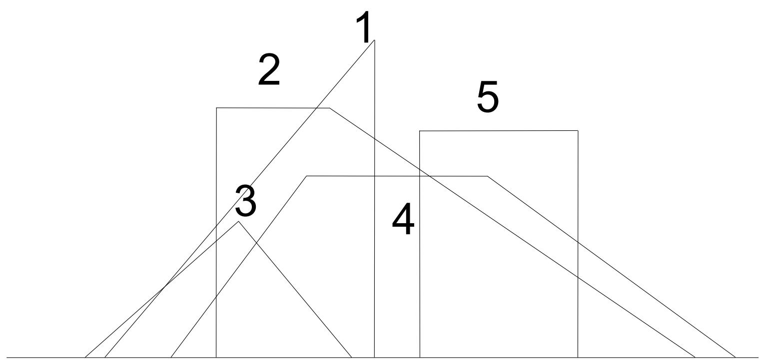
- complexity in metric constraints is due to fragmentation

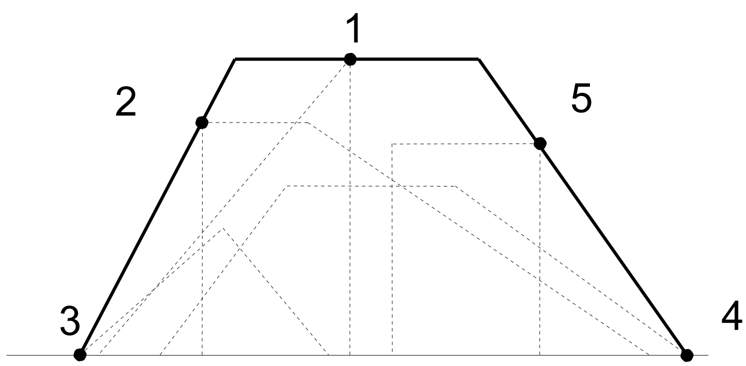
⇒ reduce fragmentation (ULT, LPC, ...)

- complexity in qualitative constraints is intrinsic in the algebra

⇒ identify new tractable sub-algebras

Fuzzy Upper-Lower Tightening (ULT^{fuz})





10.2 Fuzzy qualitative constraints

Basic principles

For complexity considerations, the concept of α-cut is useful, in fact:

- A set of fuzzy relations is tractable if all its α-cuts are classic tractable relations

- if all the classic sets coming from the α-cuts are algebras then also the original fuzzy set is an algebra

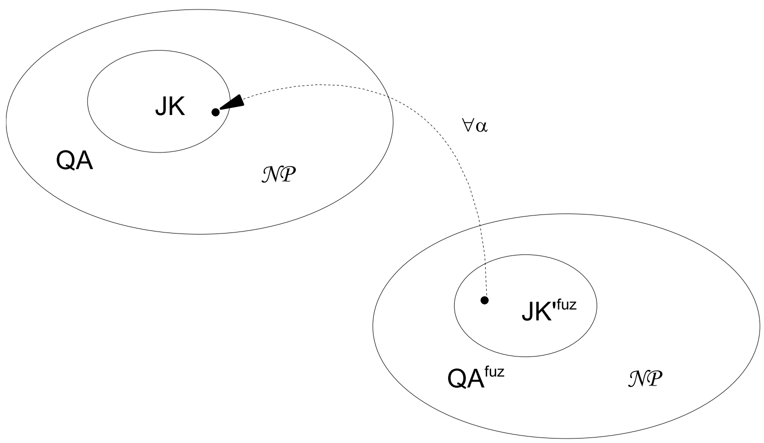
A direct application

There are 72 tractable QA fragments identified by Jonsson and Krokhin: JK

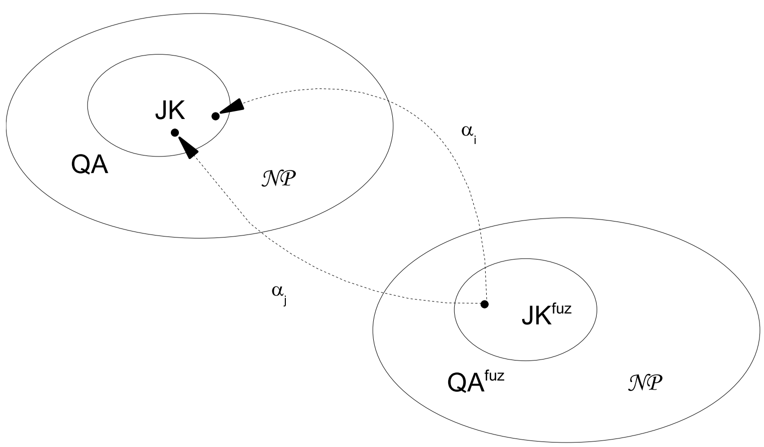
- by building the tractable fragment of QA^{fuz} in such a way that their α-cuts are in JK, the tractability can be achieved in the fuzzy case

$$JK_i'^{fuz} = \{R^{fuz} : R_\alpha^{fuz} \in JK_i, \}, i = 1 \dots 72$$

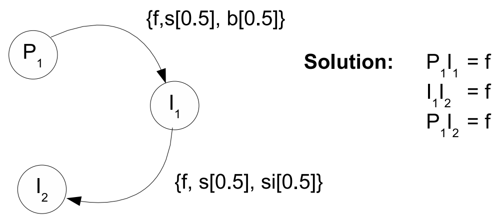
Graphical sketch



A more general definition



Example of an algebra in $JK^{fuz} \backslash JK'fuz$



To build an algebra in $JK^{fuz} \backslash JK' fuz$ we start with two α-cuts

$$R^{fuz}_{i|0.5} = \{f, s, si\} \in \mathcal{A}_1 \text{ but } \notin \mathcal{E}_p$$

and

$$R^{fuz}_{i|1.0} = \{f\} \notin \mathcal{A}_1 \text{ but } \in \mathcal{E}_p$$

Then we complete them with

$$R^{fuz}_{j|0.5} = \{f, s, b\} \in \mathcal{V}_s$$

and

$$R^{fuz}_{j|1.0} = \{f\} \in \mathcal{V}_{\mathcal{E}}$$

observing that $\mathcal{A}_1 \mathcal{V}_s = JK'_{62}$ and $\mathcal{E}_p \mathcal{V}_{\mathcal{E}} = JK'_{54}$

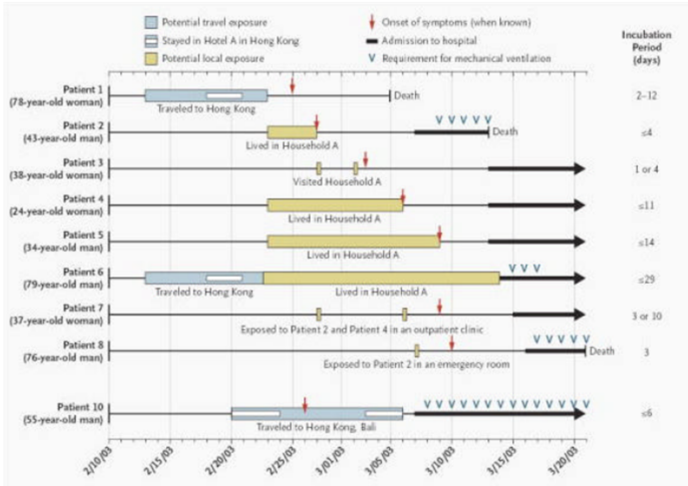
11 Applications

11.1 Medicine

Characterization of new diseases

1. Start from physician data concerning common symptoms from patients affected by an unknown disease
2. represent such data in a fuzzy constraint temporal network
3. abstract general temporal features characterizing the disease

The SARS case



Events considered

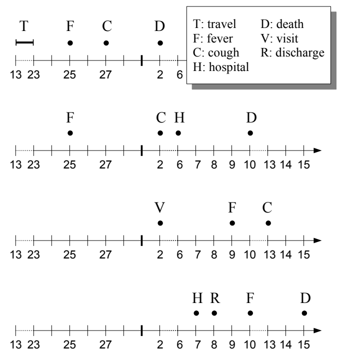
Our aim is to characterize the incubation period. To do this, we take into account:

- the period during which the disease could have been got (contagion period or CP) and its bounds
- the start of the fever
- the start of the cough
- the death

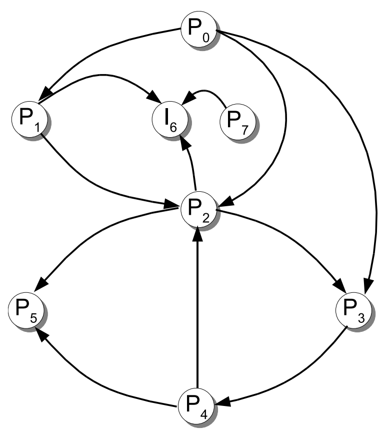
Timelines

e.g.: Patient 1:

- in travel from February 13 to February 23 (origin t0)
- 2 days later, fever
- 2 days later, cough
- 3 days later, death



The vertices



- P_0: t_0, the “origin of time”
- P_1: begin of incubation
- P_2: end of incubation
- P_3: fever
- P_4: cough
- P_5: death
- I_6: incubation period
- P_7: actual contagion

“P_i” stands for Point, “I_j” for interval

The constraints

The constraints that refer to a patient have been defined as in the following example, where we assume an uncertainty of half a day:

- about -10 days from P_0 to P_1

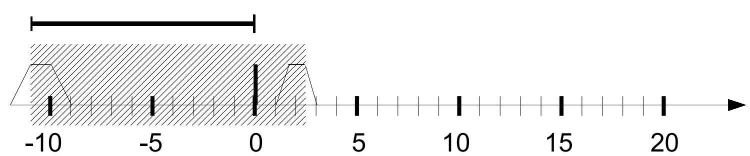
$$P_0\{[-11, -10.5, -10, -9.5]\}P_1$$

- the contagion is contained in the incubation; "s" is less plausible because the disease first has to spread in the organism

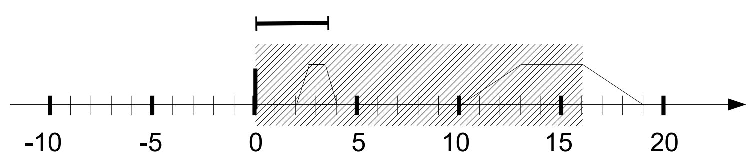
$$I_6\{d, s[0.5], f\}P_7$$

Results

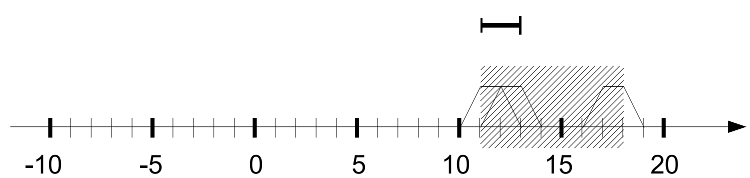
Here the hatched rectangle represents the contagion period, the interval the incubation (it ends when the first symptom appears)



about 1 to 12 days



about 0 to 4 days



about 2 to 4 days

Incubation: about 2 to 4 days

12 Extensions

Other temporal reasoning frameworks

Many extensions have been proposed, for example:

- Simple Temporal Problems with Uncertainty
- Labelled Temporal Networks
- Conditional Temporal Problems
- Simple Temporal Problems with Classes

12.1 Conditional Temporal Problems with Preferences (CTPPs)

Conditional Planning

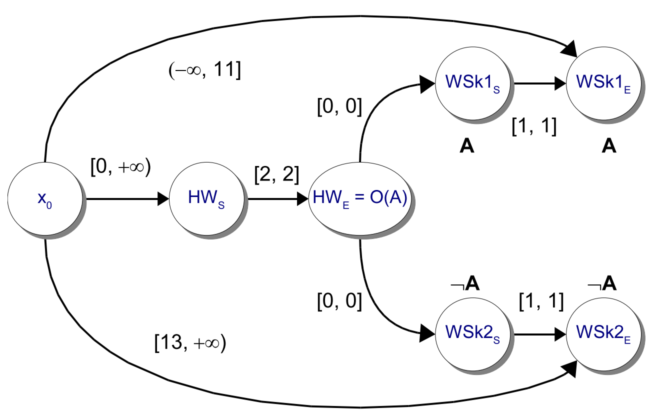
In real world a planning agent is not omniscient:

- plans cannot be generated off-line
- reactive approach is usually too restrictive (real-time requirements cannot be guaranteed)

Conditional planning adds observations actions and conditional branching

- actions are still atomic

Example of Conditional Temporal Problem



Classical $CTPs$

A CTP is a tuple $< V, E, L, OV, O, \mathcal{P} >$ where

- $\mathcal{P}$ is a set of Boolean atomic propositions $A, B, \ldots$
- V is a set of variables
- E is a set of temporal constraints between variables $v_i \in V$
- $L : V \rightarrow \mathcal{Q}^*$ is a function attaching conjunctions of literals in Q to each variable $v_i \in V$
- $OV \subseteq V$ is the set of observation variables
- $O : \mathcal{P} \rightarrow OV$ is a bijective function that associates an observation variable to a proposition. The node $O(A)$ provides the truth value for A

A variable is executed only if its associated label, i.e. a conjunction of literals, is true; once executed, it gives the truth value of the variables it observes

Consistency notions

There are three notions of consistency

- **Strong Consistency (SC):** there is a fixed way to assign values to all the variables that satisfies all projections
- **Weak Consistency (WC):** the projection of each scenario is consistent
- **Dynamic Consistency (DC):** the current partial consistent assignment can be consistently extended independently of the upcoming observations

$$SC \rightarrow DC \rightarrow WC$$

Introducing Fuzzy Rules

Labels, associated to variables, act as rules that select different execution paths

$$\text{IF } L(v) \text{ THEN EXECUTE}(v)$$

Degrees can be added

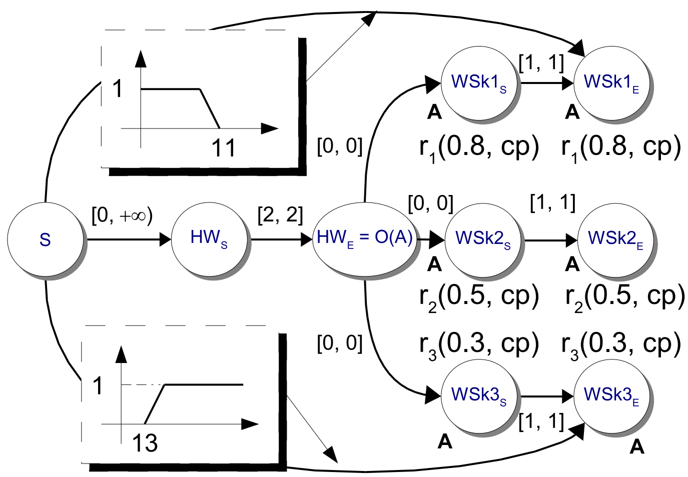
- to the premise ($pt : L(V) \rightarrow A$): truth level
- the consequence ($cp : V \rightarrow A$): preference

Formal definition of a Fuzzy $CTPP$

A $CTPP$ is a tuple $< V, E, L, OV, O, \mathcal{P} >$ where

- $\mathcal{P}$ is a finite set of **fuzzy** atomic propositions
- E is a set of **soft** temporal constraints between pairs of variables $v_i \in V$
- $L : V \rightarrow \mathcal{Q}^*$ is a function attaching conjunctions of **fuzzy** literals $\mathcal{Q} = \{p_i : p_i \in \mathcal{P}\} \cup \{\neg p_i : p_i \in \mathcal{P}\}$ to each variable $v_i \in V$
- $R : V \rightarrow \mathcal{FR}$ is a function attaching a **fuzzy rule** $r(\alpha_i, cp)$ to each variable $v_i \in V$
- $O : \mathcal{P} \rightarrow OV$ is a bijective function that associates an observation variable to each **fuzzy** atomic proposition. Variable $O(A)$ provides the **truth degree** for A.

Example of Fuzzy $CTPP$



Meta-scenarios

Scenarios in $CTPPs$ depend not only on propositions but also on threshold levels

$$\Rightarrow \text{possibly infinite}$$

- Two scenarios are equivalent is they have the same projection
- Partition scenarios in equivalence classes
- Minimal set of meta-scenarios: only one representative for each equivalence class

12.2 Fuzzy Disjoint Temporal Problems with Classes

Fuzzy metric c-constraints

Sometimes disjunctive constraints are used to model distinct scenarios which can be considered independently and which often share common parts

A **fuzzy constraint with classes**, or fuzzy c-constraint, is a constraint of the form

$$e = \{\lhd a_k, b_k, c_k, d_k \rhd [\alpha_k]_{\ll k \gg}, k \in \mathbb{N}\}$$

where k are distinct classes

$FDTPs$ with classes ($FSTP^c$)

A Fuzzy STP^c is a tuple $< V, E, M, VC, EC >$ where

- V is a set of variables
- E is a set of constraints between variables $v_i \in V$
- C is a finite set
- $VC : V \times C \rightarrow < 2^C, [0,1] >$ is a function that associates to a pair variable-class a preference
- $EC : E \times C \rightarrow < 2^C, [0,1] >$ is a function that associates to a pair constraint-class a pair of temporal bounds and a preference

Solution of a $FTDP^c$

A solution of a FDTP^c is a set of triples $< c, S, \alpha >$ where:

- c is a class
- $S : V \rightarrow \mathbb{R}$ is an assignment of the variables in V that satisfies all fuzzy constraints with class c
- α is the degree of satisfaction of the $FSTP$ associated to class c

Consistencies

There are three notions of consistency:

1. A FDTP^c is "**α-class consistent in c**" (α-CC$_c$) if the $FSTP$ associated with class c is consistent with a satisfaction degree equal to α

2. A FDTP^c is **α-existentially consistent** (α-EC) if exists a class whose associated $FSTP$ is consistent with a satisfaction degree equal to α

3. A FDTP^c is **α-universally consistent** (α-UC) if the $FSTPs$ of any class are consistent with a satisfaction degree not lower than α

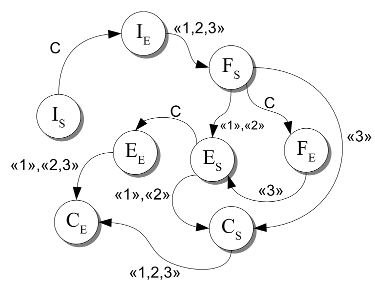
Example

Various diseases can be marked with classes

The vertices represent temporal symptoms evolutions of three diseases

- I: incubation
- F: fever
- E: exanthemata
- C: contagion

$$C = \ll 1 \gg, \ll 2 \gg, \ll 3 \gg$$



Part V
Conclusions

Towards an user-friendly integrated system